The Higgs and Coulomb/Confining Phases in "Twisted-Mass" Deformed $CP(N-1)$ Model

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Abstract

We consider non-supersymmetric two-dimensional $CP(N-1)$ model deformed by a term presenting the bosonic part of the twisted mass deformation of $\mathcal{N} = 2$ supersymmetric version of the model. Our deformation has a special form preserving a Z_N symmetry at the Lagrangian level. In the large mass limit the model is weakly coupled. Its dynamics is described by the Higgs phase, with Z_N spontaneously broken. At small masses it is in the strong coupling Coulomb/confining phase. The Z_N symmetry is restored. Two phases are separated by a phase transition. We find the phase transition point in the large-N limit. It lies at strong coupling. As was expected, the phase transition is related to broken versus unbroken Z_N symmetry in these two respective phases. The vacuum energies for these phases are determined too.

1 Introduction

As well known, two-dimensional $CP(N-1)$ model is an excellent theoretical laboratory for modeling, in a simplified environment, a variety of interesting phenomena typical of non-Abelian gauge theories in four dimensions [1, 2]. Recently, two-dimensional $CP(N-1)$ model was shown to emerge [3] as a moduli theory on the world-sheet of non-Abelian flux tubes presenting solitons in certain four-dimensional Yang–Mills theories at weak coupling [4, 5, 6, 7]. The flux tube solutions in the bulk ("microscopic") Yang–Mills theory depend on an adjustable parameter of dimension of mass. When this parameter is large the flux tubes are in fact Z_N strings; they evolve towards non-Abelian strings as the above mass parameter decreases and eventually vanishes. Correspondingly, the world-sheet theory is not just the $CP(N-1)$ model; rather it is the $CP(N-1)$ model mass-deformed in a special way that preserves a Z_N symmetry of the model. The mass term we deal with coincides with a special choice of the twisted mass [8] in supersymmetric $CP(N-1)$ model. We hasten to emphasize that we will focus exclusively on non-supersymmtric version, to which the two-phase phenomenon we study is inherent. There is no such phenomenon in supersymmetric version.

In the limit of vanishing mass deformation, the $CP(N-1)$ model is known to be a strongly coupled asymptotically free field theory [9]. A dynamical scale Λ is generated as a result of dimensional transmutation. However, at large N it can be solved by virtue of $1/N$ expansion [1]. The solution found by Witten exhibits a composite photon, coupled to N quanta n, each with charge $1/\sqrt{N}$ with respect to this photon. In two dimensions the corresponding potential is long-range. It causes linear confinement, so that only n^*n pairs show up in the spectrum. This is the reason why we refer to this phase as "Coulomb/confining." In the Coulomb/confining phase the vacuum is unique and the Z_N symmetry is unbroken.

On the other hand, if the mass deformation parameter is $\gg \Lambda$, the model is at weak coupling, the field n develops a vacuum expectation value (VEV), there are N physically equivalent vacua, in each of which the Z_N symmetry is spontaneously broken. We will refer to this regime as the Higgs phase, although this name has a Pickwick sense. Usually the Higgs mechanism implies that a gauge boson eats a would-be Goldstone meson thus acquiring a mass that screens long range interactions. In our case, at $m \gg \Lambda$, there is no gauge boson to begin with (see Sect. 4). However, the long-range interaction inherent to the Coulomb/confining phase does not take place; that's why it

is not unreasonable to refer to the phase as the Higgs phase.

In Ref. [3] it was argued that the twisted mass deformed $CP(N-1)$ model undergoes a phase transition when the value of the mass parameter is $\sim \Lambda$. The argument was largely based on analysis of the flux tubes and their evolution in the underlying four-dimensional theory. In this paper we will show, basing our consideration on two-dimensional model *per se*, in the large N limit, that a phase transition between Z_N broken and unbroken (i.e. the Higgs and Coulomb/confining) phases does indeed occur at $m^2 = \Lambda^2$. The change of regimes takes place in a narrow interval $m^2 - \Lambda^2 = O(1/N)$ where the method we use is insufficient to resolve details of the phase transition. In particular, the task of finding a conformal field theory emerging at the critical point $m^2 = \Lambda^2$ remains open.

The issue of two phases and phase transitions in related models was previously addressed by Ferrari [11, 12]. While the first paper [11] deals with $CP(N-1)$ models, neither the methods used nor results have a significant overlap with the results reported below. In [12] Ferrari exploits $1/N$ expansion methods which are similar to ours. The model to which Ref. [12] is devoted is a mass-deformed $O(N)$ model with a Z_2 symmetry at the Lagrangian level. The point of the phase transition separates the Z_2 broken phase at weak coupling from the the Z_2 unbroken phase at strong coupling, which is in parallel with our result. The phase transition in [12] is argued to be of the Ising-model type. We do not expect this to be valid in the Z_N case we deal with in the present paper.

The paper is organized as follows. In Sect. 2 we introduce the twisted mass deformation of the non-supersymmetric $CP(N-1)$ model that preserves Z_N at the Lagrangian level. We also review some well-known facts regarding this model in the strong and weak coupling regimes. The weak coupling regime (large twisted masses) is especially simple since here the theory can be treated perturbatively and exhibits a Higgs-like behavior. At strong coupling we are guided by Witten's large-N solution. In Sect. 3 large-N methods are used to solve the model at arbitrary m. The critical point separating Z_N broken and unbroken phases is determined and the vacuum energies are calculated for both phases. Section 4 presents a remark concerning inadequacy of the large-N methods for determination of the nature of the critical behavior.

2 Twisted-mass deformed $CP(N-1)$ model and its phases

In this section we describe a twisted mass deformation of the $CP(N-1)$ model preserving Z_N. Then we discuss its two distinct phases in two opposite limits, $m \ll \Lambda$ and $m \gg \Lambda$.

As was mentioned, the origin of the word "twisted" lies in supersymmetry, more exactly, extended $\mathcal{N} = 2$ supersymmetry. Aspects of the supersymmetric version were analyzed by Dorey [10], who found an exact solution in the holomorphic sector. We study the non-supersymmetric version, obtained by discarding the fermion sector.

2.1 The model

As usual in two dimensions, the Lagrangian can be cast in many different (but equivalent) forms. For our purposes the most convenient formulation is in terms of the n fields.[1] To set our notation, let us first omit the twisted mass. Then the $CP(N-1)$ model can be written as

$$S = \int d^2x \left\{ (\partial_\alpha + iA_\alpha) \, n_\ell^* \, (\partial_\alpha - iA_\alpha) \, n^\ell + \lambda \left(n_\ell^* n^\ell - r \right) \right\} , \qquad (1)$$

where n^ℓ is an N-component complex filed, $\ell = 1, 2, ..., N$, subject to the constraint

$$n_\ell^* \, n^\ell = r \qquad (2)$$

where r is the inverse coupling constant of the model. More exactly, the standard relation between r and g^2 is

$$r = 2/g^2 \, .$$

The action (1) and other similar expressions below are given in the Euclidean space. The constraint (2) is implemented by the Lagrange multiplier λ in Eq. (1). The field A_α in the Lagrangian is auxiliary too; it enters with no derivatives and can be eliminated by virtue of the equation of motion,

$$A_\alpha = -\frac{i}{2r} \, n_\ell^* \, \overleftrightarrow{\partial_\alpha} \, n^\ell \, . \qquad (3)$$

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[1] They are referred to as "quarks" or solitons in Ref. [1].

Substituting Eq. (3) in the Lagrangian, we rewrite the action in the form

$$S = \int d^2x \left\{ \partial_\alpha n_\ell^* \, \partial_\alpha n^\ell + \frac{1}{r}(n_\ell^* \partial_\alpha n^\ell)^2 + \lambda \left(n_\ell^* n^\ell - r \right) \right\} . \tag{4}$$

The model (4) is a generalization of the O(3) sigma model. The latter is formulated as

$$S = \frac{r}{4} \int d^2x \, \partial_\alpha \vec{S} \, \partial_\alpha \vec{S} \tag{5}$$

where $\vec{S}$ is a three-component vector subject to the constraint $\vec{S}^2 = 1$. At $N = 2$ the $CP(N-1)$ model (i.e. $CP(1)$) reduces to O(3) through the substitution

$$S^a = \frac{1}{r} \left(n^* \, \tau^a \, n \right), \qquad a = 1, 2, 3 , \tag{6}$$

where τ^a are the Pauli matrices. The constraint $\vec{S}^2 = 1$ follows from Eq. (2), while

$$\partial_\alpha \vec{S} \, \partial_\alpha \vec{S} \leftrightarrow \frac{4}{r} \left\{ \partial_\alpha n_\ell^* \, \partial_\alpha n^\ell + \frac{1}{r}(n_\ell^* \partial_\alpha n^\ell)^2 \right\} . \tag{7}$$

The coupling constant r is asymptotically free [9], and defines the dynamical scale of the theory Λ through

$$\Lambda^2 = M_{\text{uv}}^2 \exp\left(-\frac{4\pi r_0}{N} \right) , \tag{8}$$

where M_{uv} is the ultraviolet cut-off and r_0 is the bare coupling. The combination N/r is nothing but the 't Hooft constant which does not scale with N. As a result, Λ scales as N^0 at large N. One can also introduce the θ term, if one so desires,

$$S_\theta = \frac{i\theta}{2\pi} \int d^2x \, \varepsilon_{\alpha\gamma} \, \partial^\alpha A^\gamma = \frac{\theta}{2\pi r} \int d^2x \, \varepsilon_{\alpha\gamma} \left(\partial^\alpha n_\ell^* \, \partial^\gamma n^\ell \right) . \tag{9}$$

Now let us add to the action (1) or (4) a mass term of a special form,

$$S_m = \int d^2x \sum_\ell \left\{ n_\ell^*(\sigma^* - m_\ell^*)(\sigma - m_\ell) n^\ell \right\} , \tag{10}$$

where σ is an auxiliary complex field (with no kinetic term), and we choose

$$m_\ell = m \exp\left(\frac{2\pi i \, \ell}{N} \right), \qquad \ell = 0, 1, ..., N-1 . \tag{11}$$

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The parameter m in Eq. (11) can be assumed to be real and positive. This is not the most general choice of the twisted mass deformation. In general, a single condition is imposed, $\sum_\ell m_\ell = 0$, which destroys SU(N)/U(1) preserving only residual U$(1)^{N-1}$. We want to maintain an additional Z_N symmetry, however, which is automatic under (11). The Z_N symmetry of the action has historic roots [3], but what is important at present is that the twisted mass deformed model with Z_N symmetry is interesting on its own. The Z_N symmetry plays an important role in identifying a phase transition between the Higgs and Coulomb/confining phases of the theory.

Eliminating σ by virtue of the equation of motion,

$$\sigma = \frac{1}{r} \sum_\ell m_\ell \, n_\ell^* \, n^\ell \,, \tag{12}$$

we get

$$S_m = \int d^2x \, m^2 \left\{ r - \frac{1}{r} \left| \sum_\ell e^{\frac{2\pi i \ell}{N}} n_\ell^* \, n^\ell \right|^2 \right\} . \tag{13}$$

It is instructive to see what becomes of this mass term at $N = 2$ (i.e. the O(3) sigma model). Then, Eq. (13) implies

$$S_m = r \int d^2x \, m^2 \left(1 - S_3^2 \right), \tag{14}$$

which is obviously Z_2 symmetric. If m is large, $m \gg \Lambda$, the theory has two vacua, at $S_3 = 1$ and $S_3 = -1$. In both vacua there are two elementary excitations, S_1 and S_2, with masses $2m$.

In the general N case the action at hand has the following Z_N symmetry:

$$\sigma \rightarrow e^{i\frac{2\pi k}{N}} \sigma, \qquad n_\ell \rightarrow n_{\ell+k}$$

$$\text{for every fixed } \ell \text{ and } k = 1, 2, ..., N \,. \tag{15}$$

2.2   The Higgs phase

At large m, $m \gg \Lambda$, the renormalization group flow of the coupling constant is frozen at the scale m. Thus, the model at hand is at weak coupling and the quasiclassical analysis is applicable. The potential (10) have N

degenerate vacua which are labeled by the order parameter $\langle \sigma \rangle$, the vacuum configuration being

$$\sigma = m_{\ell_0} , \qquad n^{\ell_0} = \sqrt{r} , \quad \text{and} \quad n^\ell = 0 \text{ if } \ell \neq \ell_0 . \tag{16}$$

In each given vacuum the Z_N symmetry (15) is spontaneously broken.

There are $2(N-1)$ elementary excitations[2] with physical masses

$$M_\ell = |m_\ell - m_{\ell_0}| , \qquad \ell \neq \ell_0 . \tag{17}$$

Besides, there are kinks (domain "walls" which are particles in two dimensions) interpolating between these vacua. Their masses scale as

$$M_\ell^{\text{kink}} \sim r \, M_\ell . \tag{18}$$

The kinks are much heavier than elementary excitations at weak coupling. Note that they have nothing to do with Witten's n solitons [1] identified as solitons at strong coupling. The point of phase transition separates these two classes of solitons.

2.3   The Coulomb/confining phase

Now let us discuss the Coulomb/confining phase of the theory occurring at small m. As was mentioned, at $m = 0$ the $CP(N-1)$ model was solved by Witten in the large-N limit [1]. The model at small m is very similar to Witten's solution. (In fact, in the large-N limit it is just the same.) In Sect. 3.3 we present a generalization of Witten's analysis which we will use to study the phase transition between the Z_N asymmetric and symmetric phases. Here we just briefly summarize Witten's results for the massless model.

If $m = 0$, classically the field n^ℓ can have arbitrary direction; therefore, one might naively expect spontaneous breaking of SU(N) and the occurrence of massless Goldstone modes. Well, this cannot happen in two dimensions. Quantum effects restore the full symmetry making the vacuum unique. Moreover, the condition (2) gets in effect relaxed. Due to strong coupling we have more degrees of freedom than in the original Lagrangian, namely all N fields n become dynamical and acquire masses Λ.

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[2] Here we count real degrees of freedom. The action (1) contains N complex fields n^ℓ. The phase of n^{ℓ_0} can be eliminated from the very beginning. The condition $n_\ell^* n^\ell = r$ eliminates one more field.
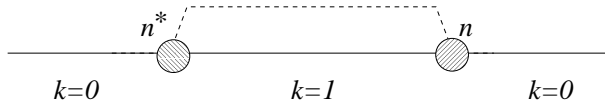
Figure 1: Linear confinement of the n-n^* pair. The solid straight line represents the ground state. The dashed line shows the vacuum energy density (normalizing $\mathcal{E}_0$ to zero).

This is not the end of the story, however. In addition, one gets another composite degree of freedom. The U(1) gauge field A_α acquires a standard kinetic term at one-loop level,[3] of the form

$$N \, \Lambda^{-2} \, F_{\alpha\beta} \, F_{\alpha\beta}. \tag{19}$$

Comparing Eq. (19) with (1) we see that the charge of the n fields with respect to this photon is $1/\sqrt{N}$. The Coulomb potential between two charges in two dimensions is linear in separation between these charges. The linear potential scales as

$$V(R) \sim \frac{\Lambda^2}{N} R \tag{20}$$

where R is separation. The force is attractive for pairs $\bar{n}$ and n, leading to the formation of weakly coupled bound states (weak coupling is the manifestation of the $1/N$ suppression of the confining potential). Charged states are eliminated from the spectrum. This is the reason why the n fields were called "quarks" by Witten. The spectrum of the theory consists of $\bar{n}n$-"mesons." The picture of confinement of n's is shown in Fig. 1.

The validity of the above consideration rests on large N. If N is not large the solution [1] ceases to be applicable. It remains valid in the qualitative sense, however. Indeed, at $N = 2$ the model was solved exactly [13, 14] (see also [15]). Zamolodchikovs found that the spectrum of the O(3) model consists of a triplet of degenerate states (with mass $\sim \Lambda$). At $N = 2$ the action (4) is built of doublets. In this sense one can say that Zamolodchikovs' solution exhibits confinement of doublets. This is in qualitative accord with the large-N solution [1].

Inside the $\bar{n}n$ mesons, we have a constant electric field, see Fig. 1. Therefore the spatial interval between $\bar{n}$ and n has a higher energy density than the domains outside the meson.

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[3]By loops here we mean perturbative expansion in $1/N$ perturbation theory.

Modern understanding of the vacuum structure of the massless $CP(N-1)$ model [16] (see also [17]) allows one to reinterpret confining dynamics of the n fields in different terms [18, 3]. Indeed, at large N, along with the unique ground state, the model has $\sim N$ quasi-stable local minima, quasi-vacua, which become absolutely stable at $N = \infty$. The relative splittings between the values of the energy density in the adjacent minima is of the order of $1/N$, while the probability of the false vacuum decay is proportional to $N^{-1}\exp(-N)$ [16, 17]. The n quanta (n quarks-solitons) interpolate between the adjacent minima.

The existence of a large family of quasi-vacua can be inferred from the study of the θ evolution of the theory. Consider the topological susceptibility, i.e. the correlation function of two topological densities

$$\int d^2x \, \langle Q(x), \; Q(0) \rangle \,, \tag{21}$$

where

$$Q = \frac{i}{2\pi}\,\varepsilon_{\alpha\gamma}\partial^\alpha A^\gamma = \frac{1}{2\pi r}\,\varepsilon_{\alpha\gamma}\left(\partial^\alpha n_\ell^* \; \partial^\gamma n^\ell\right). \tag{22}$$

The correlation function (21) is proportional to the second derivative of the vacuum energy with respect to the θ angle. From (22) it is not difficult to deduce that this correlation function scales as $1/N$ in the large N limit. The vacuum energy by itself scales as N. Thus, we conclude that, in fact, the vacuum energy should be a function of θ/N.

On the other hand, on general grounds, the vacuum energy must be a 2π-periodic function of θ. These two requirements are seemingly self-contradictory. A way out reconciling the above facts is as follows. Assume that we have a family of quasi-vacua with energies

$$E_k(\theta) \sim N\,\Lambda^2\left\{1 + \text{const}\left(\frac{2\pi k + \theta}{N}\right)^2\right\}, \quad k = 0\ldots, N-1 \tag{23}$$

A schematic picture of these vacua is given in Fig. 2. All these minima are entangled in the θ evolution. If we vary θ continuously from 0 to 2π the depths of the minima "breathe." At $\theta = \pi$ two vacua become degenerate, while for larger values of θ the former global minimum becomes local while the adjacent local minimum becomes global. It is obvious that for the neibohring vacua which are not too far from the global minimum
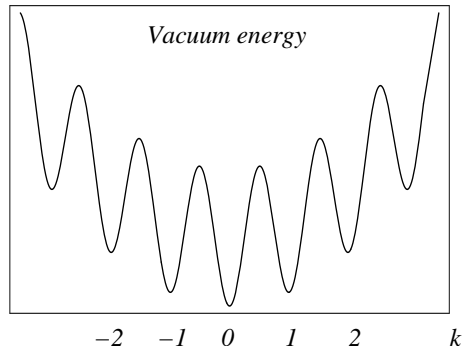
$$E_{k+1} - E_k \sim \frac{\Lambda^2}{N}\,. \tag{24}$$

Figure 2: The vacuum structure of $CP(N-1)$ model at $\theta = 0$.

This is also the confining force acting between n and $\bar{n}$.

One could introduce order parameters that would distinguish between distinct vacua from the vacuum family. An obvious choice is the expectation value of the topological charge. The kinks n^ℓ interpolate, say, between the global minimum and the first local one on the right-hand side. Then $\bar{n}$'s interpolate between the first local minimum and the global one. Note that the vacuum energy splitting is an effect suppressed by $1/N$. At the same time, kinks have masses which scale as N^0,

$$M_\ell^{n \text{ kink}} \sim \Lambda \,. \tag{25}$$

The multiplicity of such kinks is N [19], they form an N-plet of SU(N). This is in full accord with the fact that the large-N solution of (1) exhibits N quanta of the complex field n.

In summary, the $CP(N-1)$ model in the Coulomb/confining phase, at small m, has a vacuum family with a fine structure. For each given θ (except $\theta = \pi$, 3π, etc.) the true ground state is unique, but there is a large number of "almost" degenerate ground states. The splitting is of the order of Λ^2/N. The Z_N symmetry is unbroken. The spectrum of physically observable states consists of kink-anti-kink mesons which form the adjoint representation of SU(N).

At large m the theory is in the Higgs phase; it has N strictly degenerate vacua; the Z_N symmetry is broken. We have $N-1$ elementary excitations n^ℓ with masses given by Eq. (17).

Thus we conclude that these two regimes should be separated by a phase transition [3]. This phase transition is associated with the Z_N symmetry

breaking: in the Higgs phase the Z_N symmetry is spontaneously broken, while in the Coulomb phase it is restored. For $N = 2$ we deal with Z_2 which makes the situation akin to the Ising model.

3   Solution at $m \neq 0$ in the large-N limit

In this section we will generalize Witten's analysis [1] to include $m \neq 0$. The twisted mass deformed action is

$$S = \int d^2x \left\{ |\nabla_\alpha n^\ell|^2 + \lambda \left(|n^\ell|^2 - r_0 \right) + \sum_\ell |(\sigma - m_\ell)n^\ell|^2 \right\} , \qquad (26)$$

were $\nabla_\alpha = \partial_\alpha - iA_\alpha$ and m_ℓ is defined in Eq. (11), and r_0 is the bare coupling constant.

3.1   Effective theory

As soon as the action (26) is quadratic in n^ℓ we can integrate over these fields and then minimize the resulting effective action with respect to other fields. Large-N limit ensures that corrections to the saddle point approximation are small. In fact, this procedure boils down to calculating one-loop graphs with fields n^ℓ propagating inside loops.

In the Higgs phase the field n^{ℓ_0} develop a VEV. One can always choose $\ell_0 = 0$ and denote $n^{\ell_0} \equiv n$. The field n, along with σ, are our order parameters that distinguish between the Coulomb/confining and the Higgs phases, see (16).

Therefore, we do not want to integrate over n *a priori*. Instead, we will stick to the following strategy: we integrate over $N - 1$ fields n^ℓ with $\ell \neq 0$. The resulting effective action is to be considered as a functional of n, λ and σ. To find the vacuum configuration, we will minimize the effective action with respect to n, λ and σ.

Integration over n^ℓ with $\ell \neq 0$ produces the determinant

$$\prod_{\ell=1}^{N-1} \left[\det \left(-\partial_\alpha^2 + \lambda + |\sigma - m_\ell|^2 \right) \right]^{-1} , \qquad (27)$$

where we dropped the gauge field A_α. In principle, as was explained in Sect. 2.3, quasi-vacua in the Coulomb/confining phase have non-vanishing

expectation values of the operator (22). However, we cannot see these VEVs in the leading order in N. Since the analysis we carry out applies to the leading order in the large-N limit, we set $A_\alpha = 0$.

Calculating (27) we get the following contribution to the effective action:

$$\frac{1}{4\pi} \sum_{\ell=1}^{N-1} \left(\lambda + |\sigma - m_\ell|^2 \right) \left[\ln \frac{M_{\text{uv}}^2}{\lambda + |\sigma - m_\ell|^2} + 1 \right] , \qquad (28)$$

where we dropped a quadratically divergent contribution which does not depend on λ and σ.

Equation (8) implies that the bare coupling constant r_0 in (26) can be parameterized as

$$r_0 = \frac{N}{4\pi} \ln \frac{M_{\text{uv}}^2}{\Lambda^2} . \qquad (29)$$

Substituting this expression in (26) and adding (28) we see that the term proportional to $\lambda \ln M_{\text{uv}}^2$ is canceled out, and the effective action is expressed in terms of the renormalized coupling constant,

$$r_{\text{ren}} = \frac{1}{4\pi} \sum_{\ell=1}^{N-1} \ln \frac{\lambda + |\sigma - m_\ell|^2}{\Lambda^2} , \qquad (30)$$

where we neglect $O(1/N)$ contributions. In addition to the coupling constant renormalization we have to carry out renormalization of the field σ leading to a renormalization of its vacuum expectation value. To this end we add the corresponding counterterm to the bare action (26), namely,

$$-\frac{1}{4\pi} \sum_{\ell=1}^{N-1} |\sigma - m_\ell|^2 \left(\ln \frac{M_{\text{uv}}^2}{\Lambda^2} - c \right) , \qquad (31)$$

where c is a finite constant to be fixed below. This counterterm ensures that the infinite term proportional to $\sum_{\ell=1}^{N-1} |\sigma - m_\ell|^2 \ln M_{\text{uv}}^2$ in the determinant (28) is canceled and, the renormalized VEV of σ is finite. We fix the coefficient c below demanding that $\langle \sigma \rangle - m_0 = 0$ in the Higgs phase, see (16).

Assembling all contributions together we get the effective action in the form

$$S \;=\; \int d^2x \Big\{ |\partial_\alpha n|^2 + \left(\lambda + |\sigma - m_0|^2 \right) |n|^2$$

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$$+ \quad \frac{1}{4\pi} \sum_{\ell=1}^{N-1} \left(\lambda + |\sigma - m_\ell|^2\right) \left[1 - \ln \frac{\lambda + |\sigma - m_\ell|^2}{\Lambda^2}\right]$$

$$+ \quad \frac{1}{4\pi} \sum_{\ell=1}^{N-1} |\sigma - m_\ell|^2 \, c \Big\} \ . \tag{32}$$

Now, minimizing this action with respect λ, n and σ we arrive at the following set of equations:

$$|n|^2 = r_{\text{ren}} \,, \tag{33}$$

$$\left(\lambda + |\sigma - m_0|^2\right) n = 0 \,, \tag{34}$$

$$-\frac{1}{4\pi} \sum_{\ell=1}^{N-1} (\sigma - m_\ell) \ln \frac{\lambda + |\sigma - m_\ell|^2}{\Lambda^2} + (\sigma - m_0) |n|^2 + \frac{N}{4\pi} c \sigma = 0 \,, \tag{35}$$

where r_{ren} is given in Eq. (30), and we take into account the fact that the sum $\sum_{\ell=1}^{N-1} m_\ell$ is relatively suppressed: instead of $O(N)$ it is $O(1)$ so that we lose the factor of N. Equations (33), (34) and (35) represent our master set that determines the vacua of the theory. Note that Eq. (33) is a renormalized version of the bare condition (2). In addition to the above equations the vacuum configuration must satisfy two extra constraints,

$$r_{\text{ren}} \geq 0 \,, \tag{36}$$

and

$$\text{Re}\,\lambda \geq 0 \,. \tag{37}$$

The first condition just follows from $|n|^2 \geq 0$, see (33). The second one becomes clear if we examine the original bare action of the model. From Eq. (26) we conclude that the integral over λ runs along the imaginary axis (remember, we use the Euclidean formulation of the theory.) The saddle point solution for λ can have (and will have) a non-vanishing real part. To ensure convergence of the path integral over n^ℓ's the real part of λ at the saddle point should be non-negative. It is important that σ is an independent integration variable, and the integral over n^ℓ's must be convergent at all values of σ, in particular at $\sigma = m_\ell$.

We will see shortly that the constraints (36) and (37) are important conditions which single out physical phases existing in the given range of the parameter m. In the subsequent sections we will study solutions to our master-set equations and show that at large m the theory is in the Higgs phase while at small m it is in the Coulomb/confining phase, the boundary being at $m = \Lambda$.

3.2 The Higgs phase at large N

At $m \gg \Lambda$ the solution to Eqs. (33), (34) and (35) has the following form:

$$\langle \lambda \rangle = 0 \,,$$

$$\langle \sigma \rangle = m_0 \,,$$

$$\langle n \rangle = \sqrt{r_{\rm ren}} \,, \tag{38}$$

where we use the gauge freedom of the original model to choose n real, as was explained in Sect. 2.2. We see that the fields σ and n have non-vanishing VEV's and, as a result, the Z_N symmetry is spontaneously broken. Our choice $n \equiv n_0$ was of course arbitrary. In fact, we have N strictly degenerate vacua as shown in Eq. (16). Equation (35) must be used to fix the value of the constant c,

$$c = \frac{1}{N} \sum_{\ell=1}^{N-1} \left(1 - \frac{m_\ell}{m_0} \right) \ln \frac{|m_\ell - m_0|^2}{\Lambda^2} \,. \tag{39}$$

Substituting this value in the effective action (32), together with VEV's (38), we get the vacuum energy in the Higgs phase,

$$E_{\rm Higgs\ vac} = \frac{N}{2\pi}\, m^2 \,. \tag{40}$$

The logarithmic term in the second line cancels the third line.

Now, let us have a closer look at the additional constraints (36) and (37). The latter condition is trivially satisfied while to examine the impact of the condition (36) we substitute $\sigma = m_0 = m$ and $\lambda = 0$ in the expression (30) for the renormalized coupling constant. Then we get

$$r_{\rm ren} = \frac{1}{4\pi} \sum_{\ell=1}^{N-1} \ln \frac{|m_\ell - m_0|^2}{\Lambda^2} = \frac{1}{\pi} \sum_{\ell=1}^{N/2} \ln \frac{2m \sin \frac{\pi \ell}{N}}{\Lambda} = \frac{N}{2\pi} \ln \frac{m}{\Lambda} \,, \tag{41}$$

where the sum over ℓ is calculated in the large N limit. The constraint (36) implies

$$m \geq \Lambda \,. \tag{42}$$

The Higgs phase has a clear-cut meaning at large m. Hence, the above result is compatible with intuition. We will see momentarily that the lower bound of the allowed domain, $m = \Lambda$, is the phase transition point.

3.3 The Coulomb/confining phase at large N

At small m the appropriate solution of the master equations (33), (34) and (35) has the form

$$\sigma = 0 \,,$$

$$n = 0 \,,$$

$$\lambda = \Lambda^2 - m^2 \,. \tag{43}$$

The vacuum expectation value of the n field vanishes, as one would expect from the Z_N symmetric phase, while Eq. (33) is satisfied because

$$r_{\rm ren} = 0 \tag{44}$$

in the vacuum (43), cf. Eq. (30). In fact Eq. (43) becomes an $m \neq 0$ generalization of Witten's saddle point condition which was used to determine VEV of λ in [1]. Upon consulting with Eq. (26) we conclude that in our saddle point the mass of the n^ℓ quanta is Λ, independent of the value of the mass deformation parameter m. Indeed, the mass squared $\to \lambda + |\sigma - m_\ell|^2 = \Lambda^2$. Although this statement might seem counter-intuitive, it is correct. We will comment on that in the end of this section. Since both σ and n do not condense in this Coulomb/confining vacuum the Z_N symmetry is unbroken. The bare condition (2) is relaxed due to (44). The solution exhibits more degrees of freedom than are present in the Lagrangian.

Let us turn now to constraints (36) and (37). The first one is satisfied trivially while the second one implies

$$m \leq \Lambda. \tag{45}$$

We see that at $m \leq \Lambda$ the theory is in the Coulomb/confining vacuum (43) while at $m \geq \Lambda$ it is in the Higgs vacuum (38). The value
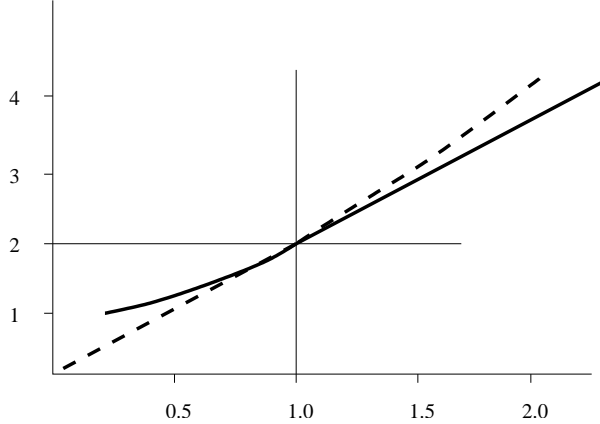
$$m_* = \Lambda \tag{46}$$

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Figure 3: Normalized vacuum energies $(4\pi E_{\text{vac}}/N\,\Lambda^2)$ versus m^2/Λ^2. The solid line shows the actual vacuum energy, while dashed lines correspond to a formal extrapolation of the Higgs and Coulomb/confinement vacuum energies to unphysical values of m below and above the phase transition point, respectively.

is the phase transition, or critical point.

Let us calculate the vacuum energy in the Coulomb phase. Substituting the vacuum values (43) in the action (32) and using expression (39) for the value of the constant c we arrive at the vacuum energy

$$E_{\text{Coulomb vac}} = \frac{N}{4\pi}\left\{\Lambda^2 + m^2 + m^2\log\frac{m^2}{\Lambda^2}\right\}, \qquad (47)$$

where the sums over ℓ are calculated in the large-N limit.

We plot the vacuum energies (47) and (40) for the Coulomb/confining and Higgs phases as a function of m^2 in Fig. 3. At the point of the phase transition (46) energy densities of both phases coincide. Moreover, their first derivatives with respect to m^2 at this point coincide too. The Higgs curve, naively extrapolated below the phase transition, runs below the Coulomb curve which might lead one to conclude that the system always stays in the Higgs phase. However, the conditions (36) and (37) produce constrains (42), (45) which tell us that at $m \leq \Lambda$ the system is in the Coulomb/confining phase while at $m \geq \Lambda$ it is in the Higgs phase.

One can check our results for the vacuum energies in both phases performing the calculations in a slightly different form, through the trace of the

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energy-momentum tensor. The vacuum energy can be obtained as

$$E_{\text{vac}} = \frac{1}{2}\langle\theta^{\mu}_{\mu}\rangle \tag{48}$$

where

$$\theta^{\mu}_{\mu} = \left[M_{\text{uv}}\partial_{M_{\text{uv}}} + \sum_{\ell=0}^{N-1}(m_l\partial_{m_\ell} + m^*_\ell\partial_{m^*_\ell}) \right] \mathcal{L}(M_{\text{uv}}, m_\ell), \tag{49}$$

and $\mathcal{L}(M_{\text{uv}}, m_l)$ is the ultraviolet-regulated Lagrangian of the model. Taking into account the classical contribution and the quantum anomaly, we precisely reproduce the vacuum energies in the Coulomb/confining and Higgs phases quoted above.

To reiterate, at large m, at weak coupling, we have N strictly degenerate vacua; the Z_N symmetry is broken. At small m, at strong coupling, a mixing between these vacua takes over, and N vacua split (see Sect. 2.3). The order parameter which marks these vacua is the VEV of the operator (22) which is non-zero for exited "vacua" with $k \neq 0$. In the leading order in N to which we are limited, we do not see this vacuum splitting. Our result exhibits a single vacuum (43). Moreover, we cannot say anything as to the nature of the phase transition. The answer can be found upon inspection of a narrow strip $|m^2 - \Lambda^2| \sim O(1/N)$ (see Sect. 4) which would require tools going beyond those exploited here.

It is curious to note that the θ dependence of physical quantities, albeit suppressed at large N, is suppressed differently above and below the critical point. If in the Higgs phase the suppression is expected to be exponential, it is power-like in the Coulomb/confining phase.

Finally, we would like to comment on the independence of the n-quanta mass on m in the Coulomb/confining phase. The crucial observation is that in the absence of n VEVs the twisted mass term (13) is actually quartic in the n fields, rather than quadratic. Therefore, while it contributes to interactions of the n fields, its contribution to the n mass cannot appear at order N^0.

4   Can we describe critical behavior?

The full solution of the phase transition problem requires establishing a conformal field theory which governs dynamics at the critical point. It is no

accident that so far nothing has been said regarding this issue. In this section we argue that the proper description of the critical behavior would require methods going beyond the $1/N$ expansion on which we rely. Thus, this question remains open.

To understand the nature of the phase transition one must identify states that become massless at the critical point $m = \Lambda$. Let us undertake this endeavor, approaching the critical point from the Higgs side.

In the Higgs phase, at large m, the theory is weakly coupled, and all excitations are massive. There are $N - 1$ complex degrees of freedom. The lowest singularity in the correlation function $\langle A_\alpha(x), \ A_\beta(0) \rangle$ is a two-particle cut, so that no stable field playing the role of a photon exist. On the other hand, at $m = 0$, there are N quanta of the U(1)-charged n fields, and a massless photon. It is natural to ask whether a light photon emerges at strong coupling as one approaches the critical point from above.

To address this issue we modify our effective action (32) including the gauge field with a kinetic term induced at one loop as in [1],

$$ S_{\text{Higgs}} = \int d^2 x \left\{ \frac{1}{4e_{\text{ren}}^2} F_{\alpha\beta}^2 + |\nabla_\alpha n|^2 + \frac{\tilde{e}_{\text{ren}}^2}{2} \left(|n|^2 - r_{\text{ren}} \right)^2 + E_{\text{Higgs vac}} \right\}, \quad (50) $$

where r_{ren} is given in Eq. (41). The kinetic term for the gauge field is induced through a loop of the n^ℓ quanta. This loop converges in the infrared domain and, as we will see shortly, is saturated by the lightest n^ℓ quanta, i.e. $\ell \sim 1$. In addition to this kinetic term, we also include a quartic term for n (we remind that $n \equiv n^0$) which comes from integration over λ in (32) around the corresponding saddle point at $\lambda = 0$ (in the quadratic approximation). The corresponding "coupling constants" denoted by e_{ren}^2 and $\tilde{e}_{\text{ren}}^2$, (in fact, they are momentum dependent; hence, the quotation marks) are

$$ e_{\text{ren}}^2 = \frac{12\pi}{\Sigma(p)}, \qquad \tilde{e}_{\text{ren}}^2 = \frac{4\pi}{\Sigma(p)}, \quad (51) $$

where

$$ \Sigma(p) = \sum_{\ell=1}^{N-1} \int \frac{dk^2}{[k^2 + |m_\ell - m_0|^2][(k-p)^2 + |m_\ell - m_0|^2]} \quad (52) $$

and p is the external momentum, $p_\alpha \leftrightarrow i\partial_\alpha$.

The lightest states in the sum (52) correspond to $\ell \sim 1$. Their masses scale as

$$ |m_\ell - m_0|^2 \sim \frac{m^2}{N^2} \quad \text{at} \quad \ell \sim 1. \quad (53) $$

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As long as the gauge field A_α and the scalar n are much heavier, they cannot be treated as stable point-like bound states.

As we reduce m, the gauge field A_α and the scalar n become lighter and eventually may cross the threshold and become genuinely stable bound states. To evaluate their masses let us take the low-energy limit in (52), assuming that p^2 is much less then the masses of lightest elementary states, see Eq. (53). Keeping only $\Sigma(0)$ we get from (50) the masses of the gauge field (the massive gauge field has one real degree of freedom) and the scalar $|n|$ (the phase of n is eaten by the Higgs mechanism), respectively,

$$m_\gamma^2 = 2\, r_{\mathrm{ren}}\, e_{\mathrm{ren}}^2\big|_{p^2=0}\ , \qquad m_n^2 = 2\, r_{\mathrm{ren}}\, \tilde{e}_{\mathrm{ren}}^2\big|_{p^2=0}\ , \tag{54}$$

$$\Sigma(0) = 0.17\,\frac{N^2}{2m^2}\ , \tag{55}$$

where we calculated the sum over ℓ in (52) at $p = 0$ numerically in the limit of large N. The renormalized coupling constant r_{ren} is given in Eq. (41).

As m approaches $m_* = \Lambda$ from above, the coupling constant r_{ren} tends to zero and, seemingly, so do the masses of the gauge field A_α and scalar n,

$$m_\gamma^2 \sim m_n^2 \sim \frac{\Lambda}{N}\,\delta m, \tag{56}$$

where $\delta m = m - m_*$. However, these conclusions would be correct only if the masses of these bound states were much smaller than the masses of the lightest n^ℓ quanta. Comparing with (53) we see that this would require

$$\frac{\delta m}{\Lambda} \ll \frac{1}{N}. \tag{57}$$

In other words, the gauge field A_α and the scalar n could become light only in a very close vicinity of the critical point. Unfortunately, in the domain (57) the expansion in $1/N$, on which we heavily rely, explodes, and we cannot trust our analysis.

Summarizing, in the narrow strip (57) near the critical point where light composite states could occur – those which could become massless at criticality – the $1/N$ expansion fails. As a result, we cannot derive the conformal field theory which would describe our system at criticality.

5   Conclusions

The mass deformed non-supersymmetric two-dimensional $CP(N-1)$ model, with a special Z_N preserving twisted mass term, surfaced recently in connection with non-Abelian strings in four-dimensional gauge theories [3]. This model turns out to be very interesting on its own, as a theory with two distinct phases and a critical point at strong coupling. Using the large-N expansion we confirmed the fact of the phase transition in m, determined the position of the critical point and calculated the vacuum energies in the Higgs and Coulomb/confining phases. The major unsolved problem is determination of the conformal field theory governing dynamics of the model at criticality.

The use of the large-N expansion allowed us to bypass such question as "what particular aspect of the strong coupling dynamics is responsible for the change of regimes at $m = m_* = \Lambda$?" Although no definite answer to this question can be given at the moment, it is tempting to conjecture that the phase transition is due to the fact that at $m < m_*$ melted instantons of Fateev et al. [20] play a crucial role while at $m > m_*$ instantons are "individualized" and suppressed.

Instantons are not suppressed at large N at $m = 0$ due to a large entropy factor. The theory can be rewritten as a massive fermion theory [20] or, equivalently, as the affine Toda theory at fixed coupling constant. Hence it is natural to ask how our large-N one-loop calculation captures nontrivial instanton effects. A possible answer can be inferred from the supersymmetric version. Supersymmetric theory can be treated in two different ways [21]. Within the first approach one exploits a similar one-loop calculation, while within the second approach summation over nonperturbative configurations yields a twisted superpotential of the affine Toda type. The mirror symmetry of supersymmetric version is responsible for equivalence of the vacuum structure in both approaches. In our non-supersymmetric version there is no evident notion of the mirror symmetry. However, one could still hope that the relation between instanton calculus and the one-loop calculation in the linear gauged formulation of $CP(N-1)$ works in a similar manner.

The phase transition at some value of the twisted mass we demonstrate in the present paper in $CP(N-1)$ seems to be a more general phenomennon taking place in a class of asymptotically free non-supersymmetric sigma models. In particular such phase transition could be expected in the Grassmannian sigma models as well as for many toric target manifolds. To an extent, these

phase transitions can be considered as non-supersymmetric counterparts of the curves of marginal stability in supersymmetric versions.

Acknowledgments

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